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Boundary singularities of solutions of N-harmonic equations with absorption

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Abstract

We study the boundary behaviour of solutions u of $-\Delta_N u + |u|^{q-1}u = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ subject to the boundary condition $u = 0$ except at one point, in the range $q > N - 1$. We prove that if $q \geqslant 2N - 1$ such an u is identically zero, while, if $N - 1 < q < 2N - 1$, u inherits a boundary behaviour which either corresponds to a weak singularity, or to a strong singularity. Such singularities are effectively constructed.

Keywords: N-Harmonic operator; Conformal invariance; Hopf lemma; Strong maximum principle

1. Introduction

Let Ω be a domain is $\mathbb{R}^N$ $(N \geqslant 2)$ with a C^2 compact boundary $\partial\Omega$. Let g be a continuous real-valued function and $a \in \partial\Omega$. This paper deals with the study of solutions $u \in C^1(\overline{\Omega} \setminus \{a\})$ of the problem

$$\begin{cases} -\operatorname{div}(|Du|^{N-2}Du) + g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{cases} \tag{1.1}$$

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and we shall be more specifically interested in the case when g has a power growth at infinity. When $N=2$, this problem falls into the scope of the boundary singularity problem for semilinear elliptic equations. The study of the N-dimensional problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{cases} \tag{1.2}$$

has been initiated by Gmira and Véron in [7]. Among the subjects under consideration were the question of removability of isolated boundary singularities and, in the case such singularities do exist, their precise description. This seminal article was at the origin of a long series of further works by Dynkin and Kuznetsov [4,5], Le Gall [11], Marcus and Véron [13] in the framework of the trace theory and, later on, the fine trace theory in the case where $g(r) = r|r|^{q-1}$, $q > 1$. One of the main reasons for such a large impact consists of the observation of the existence of a critical exponent $q = q^* = (N+1)/(N-1)$. If $q \geqslant q^*$ any solution of

$$\begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}, \end{cases} \tag{1.3}$$

is identically zero, while if $1 < q < q^*$ it appears that there exist two possible behaviours of singular solutions near a, the solutions with weak singularities and the ones with the strong singular behaviour. Later on, these two types of singular solutions played a fundamental role in the description of the rough trace of positive solutions of (1.3).

Although the techniques needed are considerably more refined, it appeared that the description of solutions of (1.1) inherits the same structure as for (1.2). The first step is to understand the model case problem

$$\begin{cases} -\operatorname{div}(|Du|^{N-2}Du) + |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}. \end{cases} \tag{1.4}$$

To this equation, we associate the homogeneous equation

$$\begin{cases} -\operatorname{div}(|Du|^{N-2}Du) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}. \end{cases} \tag{1.5}$$

It is proved in [3] that for any $k > 0$ there exists a unique solution $u = u_k$ of (1.5) satisfying

$$u_k(x) = k\frac{\rho(x)}{|x-a|^2}\bigl(1 + \mathrm{o}(1)\bigr) \quad \text{as } x \to a,\ (x-a)/|x-a| \to \sigma, \tag{1.6}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. When $k = 1$, this solution plays the role of the Poisson kernel, although neither any weak formulation nor any reasonable trace theory seems to exists, and we shall denote it by V_a^{Ω}. The behaviour (1.6) (up to a multiplicative constant) corresponds to *weak singularity behaviour* for (1.1), whenever such singularities exist. The first result we prove is the following.

Theorem. *Let $N-1<q<2N-1:=q_{\mathrm{c}}$. Then for any $k>0$ there exists a unique solution $u=u_{k,a}$ of problem* (1.4) *satisfying*

$$u_{k,a}(x)=k\frac{\rho(x)}{|x-a|^2}\big(1+\mathrm{o}(1)\big)\quad \text{as } x\to a,\ (x-a)/|x-a|\to\sigma. \tag{1.7}$$

Furthermore, $u_{\infty,a}=\lim_{k\to\infty}$ exists and is a solution of (1.4) *which satisfies*

$$\lim_{\substack{x\to a\\ \frac{x-a}{|x-a|}\to\sigma}}|x-a|^{N/(q+1-N)}u_{\infty,a}(x)=\omega(\sigma), \tag{1.8}$$

and ω is the unique positive solution of the following quasilinear equation on the upper hemisphere of the unit sphere S^{N-1},

$$\begin{cases} -\operatorname{div}_\sigma\big((\beta_q^2\omega^2+|\nabla_\sigma\omega|^2)^{(N-2)/2}\nabla_\sigma\omega\big) & \\ \quad -\Lambda(\beta_q^2\omega^2+|\nabla_\sigma\omega|^2)^{(N-2)/2}\omega+|\omega|^{q-1}\omega=0 & \text{on } S_+^{N-1},\\ \omega=0 & \text{on } \partial S_+^{N-1}, \end{cases} \tag{1.9}$$

where $\beta_q=N/(q+1-N)$ and $\Lambda=(N-1)\beta_q^2$.

The proof of the existence of $u_{k,a}$, as well as its singular behaviour, is settled upon the conformal invariance of the N-harmonic operator and the construction of subsolution of the same equation. Estimate (1.8) is proved by scaling method. The role of the critical exponent $q_{\mathrm{c}}=2N-1$ is enlighted by the following result.

Theorem. *Let g be a continuous function such that*

$$\begin{aligned} &\text{(i)}\quad \liminf_{r\to\infty} g(r)/r^{q_{\mathrm{c}}}>0,\\ &\text{(ii)}\quad \limsup_{r\to-\infty} g(r)/|r|^{q_{\mathrm{c}}}<0. \end{aligned} \tag{1.10}$$

Then any function $u\in C^1(\overline{\Omega}\setminus\{a\})$ solution of (1.1) *extends as a function $\tilde u\in C(\overline{\Omega})$.*

As in the semilinear case, the occurrence coincides with the case where the blow-up exponent $-\beta_q$ which is natural for Eq. (1.4) coincides with the one of the function V_a^Ω solution of (1.5). Finally we provide the full classification of positive solutions of problem (1.4).

Theorem. *Let $N-1<q<q_{\mathrm{c}}$ and u is any nonnegative solution of* (1.4)*, then*

(i) *either $u\equiv 0$,*
(ii) *either there exists $k>0$ such that $u=u_{k,a}$,*
(iii) *or $u=u_{\infty,a}$.*

In the proof of (iii) the boundary Harnack inequalities that satisfies any positive solution of (1.4) (see [1]) play a fundamental role. The role of Harnack inequalities has already been a key tool for studying internal isolated singularities for related equations (see [6,14,15,20]).

Our paper is organized as follows. Section 1—Introduction, Section 2—Weak and strong boundary singularities, Section 3—The removability result, Section 4—The classification theorem.

2. Weak and strong boundary singularities

The construction of positive solutions of

$$-\operatorname{div}\big(|Du|^{N-2}Du\big)+|u|^{q-1}u=0, \tag{2.1}$$

is settled upon three facts: the existence of solutions to the homogeneous equation

$$-\operatorname{div}\big(|Du|^{N-2}Du\big)=0, \tag{2.2}$$

the conformal invariance of (2.2) and an a priori estimate satisfied by any solution of (2.1). Throughout this paper C denotes a positive constant which depends only on the structural assumptions corresponding to N, p, q and Ω. The value of the constant may change from one occurrence to another.

Proposition 2.1. *Let $\Omega\subset\mathbb{R}^N$ be a domain with a compact boundary and $a\in\partial\Omega$. Consider real numbers $q>p-1>0$, $A>0$ and $B\geqslant 0$. If $u\in C(\overline{\Omega}\setminus\{a\})\cap W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of*

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du)+A|u|^{q-1}u\leqslant B & \text{in } \Omega,\\ u\leqslant 0 & \text{on } \partial\Omega\setminus\{a\}, \end{cases} \tag{2.3}$$

it satisfies

$$u(x)\leqslant\left(\frac{\lambda}{A|x-a|^p}\right)^{1/(q+1-p)}+\left(\frac{\mu B}{A}\right)^{1/q}\quad \forall x\in\overline{\Omega}\setminus\{a\}, \tag{2.4}$$

where λ and μ depends on N, p and q.

Proof. By assumption

$$\int_\Omega\big(|Du|^{p-2}Du\,.\,D\zeta+A|u|^{q-1}u\zeta\big)\,dx\leqslant B\int_\Omega\zeta\,dx \tag{2.5}$$

for any $\zeta\in W^{1,p}(\Omega)$ with compact support, $\zeta\geqslant 0$. Let $\eta\in C^2(\mathbb{R})$ be a nonnegative function such that $0\leqslant\eta'\leqslant 1$, $\eta''\geqslant 0$, $\eta=\eta'=\eta''$ on $(-\infty,0]$, $0<\eta(r)\leqslant r$ on $(0,\infty)$. For $\epsilon>0$ we set $\eta_\epsilon(r)=\eta((r-\epsilon)_+)$. Let $\zeta\in W^{1,p}(\mathbb{R}^N\setminus\{0\})$ with compact support. Inasmuch $(\eta'_\epsilon(u))^{p-1}\zeta$ has compact support in Ω and

$$D\big(\big(\eta'_\epsilon(u)\big)^{p-1}\zeta\big)=\big(\eta'_\epsilon(u)\big)^{p-1}D\zeta+(p-1)\big(\eta'_\epsilon(u)\big)^{p-2}\eta''_\epsilon(u)\zeta\,Du,$$

it belongs to $W^{1,p}(\Omega)$ and is an admissible test function for (2.5). Thus

$$\int_\Omega \big(|Du|^{p-2}Du\,.\,D\big(\big(\eta'_\epsilon(u)\big)^{p-1}\zeta\big)+A|u|^{q-1}u\big(\eta'_\epsilon(u)\big)^{p-1}\zeta\big)\,dx \leqslant B\int_\Omega \big(\eta'_\epsilon(u)\big)^{p-1}\zeta\,dx,$$

and

$$|Du|^{p-2}Du\,.\,D\big(\big(\eta'_\epsilon(u)\big)^{p-1}\zeta\big) \geqslant \big(\eta'_\epsilon(u)\big)^{p-1}|Du|^{p-2}Du\,.\,D\zeta = |Dv_\epsilon|^{p-2}\,Dv_\epsilon\,.\,D\zeta,$$

where we have set $v_\epsilon = \eta_\epsilon(u)$. Furthermore, η can be chosen such that $r^q(\eta'_\epsilon(r))^{p-1} \geqslant \eta^q_\epsilon(r)$, for example, if we fix $\eta(r)=r^2/2\delta$ on $(0,\delta]$ and $\eta(r)=r-\delta/2$ on $[\delta,\infty)$ for some $\delta>0$. We extend v_ϵ by 0 outside $\overline{\Omega}\setminus\{a\}$ and denote by $\tilde v_\epsilon$ the new function, then $\tilde v_\epsilon \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{a\}) \cap C(\mathbb{R}^N\setminus\{a\})$ and

$$\int_\Omega \big(|D\tilde v_\epsilon|^{p-2}\,D\tilde v_\epsilon\,.\,D\zeta + A|\tilde v_\epsilon|^{q-1}\tilde v_\epsilon\zeta\big)\,dx \leqslant B\int_\Omega \zeta\,dx. \tag{2.6}$$

This means that $\tilde v_\epsilon$ is a weak subsolution in $\mathbb{R}^N\setminus\{a\}$. By [18, Lemma 1.3], we derive

$$\tilde v_\epsilon(x) \leqslant \left(\frac{\lambda}{A|x-a|^p}\right)^{1/q+1-p} + \left(\frac{\mu B}{A}\right)^{1/q} \quad \forall x\in\mathbb{R}^N\setminus\{a\},$$

for some $\lambda>0$ and $\mu>0$ depending on N, p and q. Letting successively $\epsilon\to 0$ and $\delta\to 0$ we obtain (2.3). □

When Ω is smooth we have a sharper estimate.

Proposition 2.2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^2 boundary and $a\in\partial\Omega$. Let $q\geqslant p-1>1$ and $A>0$. If $u\in C(\overline{\Omega}\setminus\{a\})\cap W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of* (2.3) *with $B=0$, there exists $C>0$ depending on Ω, p and q such that*

$$u(x) \leqslant \frac{C\rho(x)}{(A|x-a|^{q+1})^{1/(q+1-p)}} \quad \forall x\in\overline{\Omega}\setminus\{a\}, \tag{2.7}$$

where $\rho(x)=\operatorname{dist}(x,\partial\Omega)$.

Proof. By translation we can assume that $a=0$. For $\epsilon>0$ let v_ϵ be the solution of

$$\begin{cases} -\operatorname{div}(|Dv_\epsilon|^{p-2}Dv_\epsilon)+A|v|^{q-1}_\epsilon v_\epsilon = 0 & \text{in } \Omega^\epsilon=\Omega\setminus B_\epsilon,\\ v_\epsilon = u_+ & \text{on } \partial\Omega^\epsilon. \end{cases} \tag{2.8}$$

By [18, Lemma 1.3] as in the proof of Proposition 2.1 and the maximum principle, there holds

$$u_+(x) \leqslant v_\epsilon(x) \leqslant \left(\frac{\lambda}{A(|x|-\epsilon)^p}\right)^{1/(q+1-p)} \quad \forall x\in\overline{\Omega}^\epsilon.$$

Consequently $\epsilon \leqslant \epsilon' \Rightarrow v_\epsilon \geqslant v_{\epsilon'}$. Letting $\epsilon \to 0$ and using the previous inequalities and the classical regularity results for solutions of quasilinear equations [12] we conclude that v_ϵ converges, as $\epsilon \to 0$, to some v which is a nonnegative solution of

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) + Av^q = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases} \tag{2.9}$$

and dominate u. Further, if $\ell > 0$ the function v^ℓ defined by $v^\ell(y) = \ell^{p/(q+1-p)} v(\ell y)$ is a solution of (2.9) with Ω replaced by $\Omega_\ell = \ell^{-1}\Omega$. Let $x \in \overline{\Omega} \setminus \{0\}$ and $\ell = |x|$. Since

$$0 \leqslant v^\ell(y) \leqslant \left(\frac{\lambda}{A(|y|)^p}\right)^{1/(q+1-p)} \quad \forall y \in \Omega_\ell,$$

and

$$\max\{|Dv^\ell(y)|:\ y \in \Omega_\ell \cap B_{3/2} \setminus B_{2/3}\} \leqslant M \max\{|v^\ell(z)|:\ z \in \Omega_\ell \cap B_2 \setminus B_{1/2}\},$$

where M is uniformly bounded because the curvature of $\partial\Omega_\ell$ is bounded, we obtain that $Dv^\ell(y)$ is uniformly bounded by some constant C on $\Omega_\ell \cap B_{3/2} \setminus B_{2/3}$. Because $Dv^\ell(y) = \ell^{(q+1)/(q+1-p)} Dv(\ell y)$, it follows that

$$|Dv(x)| \leqslant \frac{C}{A^{1/q+1-p}|x|^{(q+1)/(q+1-p)}}.$$

By the mean value theorem, and using the fact that v vanishes on $\partial\Omega \setminus \{0\}$, we derive

$$v(x) \leqslant \frac{C\rho(x)}{A^{1/q+1-p}|x|^{(q+1)/(q+1-p)}},$$

which implies (2.7). □

The construction of solutions of the quasilinear equations (2.1) with prescribed isolated singularity on the boundary of a general C^2 bounded domain Ω is settled upon similar constructions when the domain is either a half space, or a ball.

Proposition 2.3. *Assume $N - 1 < q < 2N - 1$ and let $H = \mathbb{R}^N_+ = \{x = (x_1, \dots, x_N):\ x_N > 0)\}$ and $k > 0$. Then there exists a unique positive solution $u = u_k^H \in C^1(\overline{H} \setminus \{0\})$ of (2.1) in H which vanishes on $\partial H \setminus \{0\}$ and satisfies,*

$$u(x) = k\frac{x_N}{|x|^2}\big(1 + \mathrm{o}(1)\big) \quad \text{as } x \to 0. \tag{2.10}$$

Proof. Since the function $x \mapsto kx_N|x|^{-2}$ is N-harmonic in H and vanishes on $\partial H \setminus \{0\}$, it is a supersolution of (2.1). We write spherical coordinates in $\mathbb{R}^N$ under the form

$$x = \{(r, \sigma) \in [0, \infty) \times S^{N-1} = (r, \sin\phi\sigma', \cos\phi):\ \sigma' \in S^{N-2},\ \phi \in [0, \pi]\}, \tag{2.11}$$

then

$$Du = u_r\mathbf{i} + \frac{1}{r}\nabla_\sigma u,$$

where $\mathbf{i} = x/|x|$, ∇_σ denotes the covariant gradient on S^{N-1}, and Eq. (2.1) takes the form

$$\begin{aligned}&-r^{1-N}\big(r^{N-1}\big(u_r^2 + r^{-2}|\nabla_\sigma u|^2\big)^{(N-2)/2}u_r\big)_r\\ &\quad - r^{-2}\operatorname{div}_\sigma .\big(\big(u_r^2 + r^{-2}|\nabla_\sigma u|^2\big)^{(N-2)/2}\nabla_\sigma u\big) + |u|^{q-1}u = 0.\end{aligned} \tag{2.12}$$

Next

$$\nabla_\sigma u = -u_\phi \mathbf{e} + \frac{1}{\sin\phi}\nabla_{\sigma'}u,$$

where $\mathbf{e}$ is derived from $x/|x|$ by the rotation with angle $\pi/2$ in the plane $0, x, N$ (N being the North pole), and $\nabla_{\sigma'}$ is the covariant gradient on S^{N-2} and (see [3])

$$\begin{aligned}&\operatorname{div}_\sigma .\left(\left(u_r^2 + \frac{|\nabla_\sigma u|^2}{r^2}\right)^{(N-2)/2}\nabla_\sigma u\right)\\ &\quad = \frac{1}{\sin^{N-2}\phi}\left(\sin^{N-2}\phi\left(u_r^2 + \frac{u_\phi^2}{r^2} + \frac{|\nabla_{\sigma'}u|^2}{r^2\sin^2\phi}\right)^{(N-2)/2}u_\phi\right)_\phi\\ &\qquad + \frac{1}{\sin^2\phi}\operatorname{div}_{\sigma'}\left(\left(u_r^2 + \frac{u_\phi^2}{r^2} + \frac{|\nabla_{\sigma'}u|^2}{r^2\sin^2\phi}\right)^{(N-2)/2}\nabla_{\sigma'}u\right).\end{aligned} \tag{2.13}$$

If u depends only on r and ϕ, (2.1) takes the form

$$\begin{aligned}&-r^{1-N}\big(r^{N-1}\big(u_r^2 + r^{-2}u_\phi^2\big)^{(N-2)/2}u_r\big)_r\\ &\quad - r^{-2}\sin^{2-N}\phi\big(\sin^{N-2}\phi\big(u_r^2 + r^{-2}u_\phi^2\big)^{(N-2)/2}u_\phi\big)_\phi + |u|^{q-1}u = 0.\end{aligned} \tag{2.14}$$

Step 1. We look for a local subsolution w under the form

$$w(r,\sigma) = k\big(1 - r^\alpha\big)r^{-1}\cos\phi, \quad r > 0,\ \phi \in [0, \pi/2],$$

where $\alpha > 0$ is to be determined. Then

$$\begin{gathered}w_r = -kr^{-2}\big(1 + (\alpha-1)r^\alpha\big)\cos\phi \quad \text{and} \quad w_\phi = -kr^{-1}\big(1 - r^\alpha\big)\sin\phi,\\ w_r^2 + r^{-2}w_\phi^2 := P = k^2r^{-4}\big(1 + 2\big(\alpha\cos^2\phi - 1\big)r^\alpha + r^{2\alpha}\big(\big(\alpha^2 - 2\alpha\big)\cos^2\phi + 1\big)\big),\\ w_{rr} = kr^{-3}\big(2 - (\alpha-1)(\alpha-2)r^\alpha\big)\cos\phi \quad \text{and} \quad w_{\phi\phi} = -kr^{-1}\big(1 - r^\alpha\big)\cos\phi,\\ P_r = -2k^2r^{-5}\big[2 + (4-\alpha)\big(\alpha\cos^2\phi - 1\big)r^\alpha + (2-\alpha)\big(\big(\alpha^2 - 2\alpha\big)\cos^2\phi + 1\big)r^{2\alpha}\big],\\ P_\phi = -k^2\alpha r^{\alpha-4}\big[2 + (\alpha-2)r^\alpha\big]\sin 2\phi,\end{gathered}$$

$$P_r w_r + r^{-2} P_\phi w_\phi = 2k^3 r^{-7}\big[2 + \big(5\alpha - 6 + \big(2\alpha - \alpha^2\big)\cos^2\phi\big)r^\alpha + O\big(r^{2\alpha}\big)\big]\cos\phi,$$

$$(N-1)r^{-1}w_r + w_{rr} + (N-2)r^{-2}\cot\phi w_\phi + r^{-2}w_{\phi\phi}$$
$$= kr^{-3}\big[4 - 2N + (2-\alpha)(N+\alpha-2)r^\alpha\big]\cos\phi.$$

Since

$$\begin{aligned}
&-\operatorname{div}\big(|Dw|^{N-2}Dw\big) + w^q \\
&\quad = Lw \\
&\quad = -P^{(N-2)/2}\big[(N-1)r^{-1}w_r + w_{rr} + (N-2)r^{-2}\cot\phi w_\phi + r^{-2}w_{\phi\phi}\big] \\
&\qquad - \frac{N-2}{2}P^{(N-4)/2}\big[P_r w_r + r^{-2}P_\phi w_\phi\big] + w^q,
\end{aligned}$$

and

$$w^q = k^q\big(1 - r^\alpha\big)^q r^{-q}\cos^q\phi = k^q\big(1 - qr^\alpha + O\big(r^{2\alpha}\big)\big)r^{-q}\cos^q\phi,$$

a straightforward computation leads to

$$\begin{aligned}
Lw &= k^{p-1}\alpha\big[3 - 2N + (2+\alpha)(N-2)\cos^2\phi + O\big(r^\alpha\big)\big]P^{(N-4)/2}r^{\alpha-7}\cos\phi \\
&\quad + k^q\big(1 - qr^\alpha + O\big(r^{2\alpha}\big)\big)r^{-q}\cos^q\phi \\
&= k^{p-1}\alpha\big[3 - 2N + (2+\alpha)(N-2)\cos^2\phi\big]r^{-(2N-1)+\alpha}\cos\phi + k^q r^{-q}\cos^q\phi \\
&\quad - qk^q r^{-q+\alpha}\cos^q\phi + O\big(r^{-(2N-1)+2\alpha}\cos\phi\big) + O\big(r^{-q+2\alpha}\cos\phi\big).
\end{aligned} \tag{2.15}$$

By assumption $q < 2N-1$. If we choose $\alpha < \min\{2N-1-q, 1/(N-2)\}$, there exists $R \in (0,1]$ such that $Lw \leqslant 0$ on $H \cap B_R$.

Step 2. Next we construct a solution u_R in $B_R \cap H$ which vanishes on $\partial B_R \cap H$ and on $\partial H \setminus \{0\}$ and satisfies

$$\lim_{r\to 0}\frac{ru_R(r,\sigma)}{\cos\phi} = k. \tag{2.16}$$

Let $\ell_R = k(1 - R^\alpha)R^{-1}$. Inasmuch $w - \ell_R$ is a subsolution, for any $\epsilon > 0$ we can construct a nonnegative solution u_ϵ of (2.1) in $H \cap (B_R \setminus B_\epsilon)$ which vanishes on $H \cap \partial B_R$ and on $\partial H \cap (B_R \setminus B_\epsilon)$ and takes the value $k\epsilon^{-2}x_N$ on $H \cap \partial B_\epsilon$. By comparison

$$\big(w(x) - \ell_R\big)_+ \leqslant u_\epsilon(x) \leqslant kx_N|x|^{-2}. \tag{2.17}$$

Furthermore, $\epsilon \mapsto u_\epsilon$ is increasing. Set $u = u_R = \lim_{\epsilon\to 0}u_\epsilon$, then u is a solution of (2.1) in $H \cap B_R$ which vanishes on $\partial B_R \cap H$ and on $\partial H \setminus \{0\}$ and satisfies the same inequality (2.17) as u_ϵ, but in whole $H \cap B_R$. This implies that (2.16) holds uniformly on $[0, \pi/2 - \delta]$, for any

$\delta > 0$. In order to improve this inequality, we perform a scaling: for $r > 0$, we set $u^r(x) = ru(rx)$. Then u^r satisfies

$$-\operatorname{div}\big(|Du^r|^{N-2}Du^r\big) + r^{2N-1-q}\big(u^r\big)^q = 0 \tag{2.18}$$

in $H \cap B_{R/r}$ where there holds

$$k\big(x_N|x|^{-2}\big(1 - r^\alpha|x|^\alpha\big) - \ell_R\big)_+ \leqslant u^r(x) \leqslant kx_N|x|^{-2}. \tag{2.19}$$

Since u^r is uniformly bounded for $1/2 \leqslant |x| \leqslant 2$, it follows from regularity theory [12] that it is also bounded in the $C^{1,\alpha}$-topology of $2/3 \leqslant |x| \leqslant 3/2$. Using Ascoli's theorem and the fact that $u^r(x)$ converges to $kx_N|x|^{-2}$ pointwise and locally uniformly, it follows that $Du^r(x) = r^2Du(rx)$ converges uniformly in $\{x \in H\colon 2/3 \leqslant |x| \leqslant 3/2\}$ to $-2kx_N|x|^{-4}x + k|x|^{-2}\mathbf{e}_N$ which is the gradient of $x \mapsto kx_n|x|^{-2}$. Using the expression of Du in spherical coordinates we obtain

$$r^2u_r\mathbf{i} - ru_\phi\mathbf{e} + \frac{r}{\sin\phi}\nabla_{\sigma'}u \to -2k\sigma_N\mathbf{i} + k\mathbf{e}_N \quad \text{uniformly on } S_+^{N-1} \text{ as } r \to 0,$$

where $\sigma_N = \langle\sigma, \mathbf{e}_N\rangle$. Inasmuch $\mathbf{i}$, $\mathbf{e}$ and $\nabla_{\sigma'}u$ are orthogonal, the component of $\mathbf{e}_N$ is $\sin\phi$, thus

$$ru_\phi(r, \sigma', \phi) \to -k\sin\phi \quad \text{as } r \to 0. \tag{2.20}$$

Since

$$u(r, \sigma', \phi) = \int_{\pi/2}^{\phi} u_\phi(r, \sigma', \theta)\,d\theta, \tag{2.21}$$

the previous convergence estimate establishes (2.16).

Step 3. *Construction of the solution in H.* Let η be the truncation function introduced in the proof of Proposition 2.1, and $\eta_\epsilon(r) = \eta((r-\epsilon)_+)$. Then the function $u_{R,\epsilon}$ defined by $u_{R,\epsilon} = \eta_\epsilon \circ u_R$ in $H \cap B_R$ and zero outside, is a subsolution of (2.1) in H which vanishes on $\partial H \setminus \{0\}$ and satisfies (2.16). Using the same device as in Step 2, we construct a sequence of solutions u_δ ($\delta > 0$) of (2.1) in $H \setminus B_\delta$ with boundary value $k\delta^{-2}x_N$ on $\partial B_\delta \cap H$, zero on $\partial H \setminus B_\delta$ and satisfies

$$u_{R,\epsilon} \leqslant u_\delta \leqslant kx_N|x|^{-2}.$$

When $\delta \to 0$, u_δ decreases and converges to some u which satisfies (2.1) and the previous inequality. Letting successively $\epsilon \to 0$ and $\eta(r) \to r_+$ we obtain that u satisfies

$$\check{u}_R(x) \leqslant u(x) \leqslant kx_N|x|^{-2} \quad \text{in } H, \tag{2.22}$$

where $\check{u}$ is the extension of u by zero outside B_R. The proof of (2.10) is the same as in Step 2.

Step 4. *Uniqueness*. Let u and $\hat{u}$ be two solutions of (2.1) satisfying (2.10) and $\epsilon > 0$. Then $u_\epsilon = (1+\epsilon)u + \epsilon$ is a supersolution which is positive of $\partial H \setminus \{0\}$. Inasmuch it dominates $\hat{u}$ both in a neighborhood of 0 and in a neighborhood of infinity, it dominates $\hat{u}$ in H. Letting $\epsilon \to 0$ yields to $u \geqslant \hat{u}$. Similarly $\hat{u} \geqslant u$. □

Proposition 2.4. *Assume $N-1 < q < 2N-1$ and let $B = B_1(0)$, $a \in \partial B$ and $k > 0$. Then there exists a unique function $u = u^B_{k,a} \in C^1(\overline{B} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies* (2.1) *in B and*

$$u(x) = k\frac{1-|x|}{|x-a|^2}\big(1+\mathrm{o}(1)\big) \quad \textit{as } x \to a. \tag{2.23}$$

Proof. With a change of coordinates, we can assume that B has center $m = (0,\dots,0,-1/2)$ and a is the origin of coordinates. We denote by ω the point $(0,\dots,0,-1)$ and by $\mathcal{I}_\omega$ the inversion with center ω and power 1. By this involutive transformation, the half-space $H = \{x \in \mathbb{R}^N\colon x_N > 0\}$ is transformed into the ball $B^* = \{x \in \mathbb{R}^N\colon |x|^2 + x_N < 0\}$. Thus the function $x \mapsto P_k(x) = -k(|x|^2 + x_N)/2|x|^2$ is N-harmonic and positive in B^*, vanishes on $\partial B^* \setminus \{0\}$ and is singular at 0. Let v_k be the solution of (2.1) in H satisfying (2.10), and $u_k = v_k \circ \mathcal{I}_\omega$. Then $u_k \in C(\overline{B^*} \setminus \{0\})$ satisfies

$$\begin{cases} -\operatorname{div}(|Du_k|^{N-2}Du_k) + |x-\omega|^{-2N}u_k^q = 0 & \text{in } B^*, \\ u_k = 0 & \text{on } \partial B^* \setminus \{0\}. \end{cases} \tag{2.24}$$

Furthermore, $u_k \leqslant P_k$ and

$$P_k(x) = k\frac{1/4 - |x-m|^2}{2|x|^2} = k\frac{1/2 - |x-m|}{2|x|^2}\big(1+\mathrm{o}(1)\big) = u_k(x)\big(1+\mathrm{o}(1)\big) \tag{2.25}$$

as $x \to 0$. Inasmuch $|x-\omega| \leqslant 1$, u_k is a subsolution of (2.1) in B^*. For $\epsilon > 0$ we construct a solution v_ϵ of (2.1) in $B^* \setminus B_\epsilon(0)$ with boundary value P_k. By the maximum principle $u_k \leqslant v_\epsilon \leqslant P_k$ in $B^* \setminus B_\epsilon(0)$. Since the sequence $\{v_\epsilon\}$ is monotone, we obtain that there exists a solution $\lim_{\epsilon\to 0} v_\epsilon := u \in C^1(\overline{B^*} \setminus \{0\})$ of (2.1) in B^* which satisfies

$$u_k(x) \leqslant u(x) \leqslant P_k(x) \quad \text{in } B^*, \tag{2.26}$$

and

$$u(x) = k\frac{1/2 - |x-m|}{2|x|^2}\big(1+\mathrm{o}(1)\big). \tag{2.27}$$

We change the variables in setting $x'_N = x_N + 1/2$ and $x'_i = x_i$ $(i = 1,\dots,N-1)$. We define $u'(x') = u(x)$ and denote by a the point $(0,\dots,0,1)$. Clearly u' satisfies (2.1) in $B_{1/2}$, vanishes on $\partial B_{1/2} \setminus \{a\}$ and

$$u'(x) = k\frac{1/2 - |x|}{2|x-a/2|^2}\big(1+\mathrm{o}(1)\big) \quad \text{as } x \to a/2. \tag{2.28}$$

By the transformation $\ell \mapsto \ell^{p/(q+1-p)} u_k'(\ell x)$, where $\ell = 1/2$, we obtain a solution $u_{k,a}$ of (2.1) in B which verifies

$$u_{k,a}(x) = 2^{N/(q+1-N)} k \frac{1-|x|}{|x-a|^2}\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } x \to a. \tag{2.29}$$

Because k is arbitrary, (2.23) follows. Uniqueness of the solution is obtained as in Proposition 2.3 with $u_\epsilon = (1+\epsilon)u$. $\square$

Proposition 2.5. *Assume $N-1 < q < 2N-1$ and let $G = \overline{B}^{\mathrm{c}}$, $a \in \partial B$ and $k > 0$. Then there exists a unique function $u = u_{k,a}^{B^{\mathrm{c}}} \in C^1(\overline{G} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies* (2.1) *in G and*

$$u(x) = k \frac{|x|-1}{|x-a|^2}\bigl(1+\mathrm{o}(1)\bigr) \quad \textit{as } x \to a. \tag{2.30}$$

Proof. Uniqueness follows from (2.30) by the same method as in Propositions 2.3 and 2.4. Actually, it will be proved in Theorem 2.7. For existence we perform the inversion $\mathcal{I}_0^1$ with center 0 and power 1. It transforms the function $u_{k,a}^B$ constructed in the previous proposition into a function $v \in C^1(\overline{G} \setminus \{a\})$ which vanishes on $\partial B \setminus \{a\}$ and satisfies (2.30). Furthermore, v is solution of

$$-\operatorname{div}\bigl(|Dv|^{N-2} Dv\bigr) + |x|^{-2N} |v|^{q-1} v = 0 \tag{2.31}$$

in G. Since $|x| > 1$, v is a supersolution for (2.1) in G. With no loss of generality, we can assume that $a = (0,\dots,0,1)$ and let $u_{k,a}^{H^1}$ be the solution of (2.1) in $H^1 = \{x = (x_1,\dots,x_N\colon x_N > 1)\}$ satisfying (2.10) already constructed in Proposition 2.3. Then $\upsilon_\epsilon = \eta(u_{k,a}^{H^1})$ is a subsolution in G (where η_ϵ has been defined in the proof of Proposition 2.1). By the same approximation as in the previous proposition, we construct an increasing sequence $\{u_\epsilon\}$ ($\epsilon > 0$) of solutions of (2.1) in $G \setminus B_\epsilon(a)$ which vanishes on $\partial G \setminus B_\epsilon(a)$, takes the value v on $G \cap \partial B_\epsilon(a)$ and verifies $\upsilon_\epsilon \leqslant u_\epsilon \leqslant v$ in $G \setminus B_\epsilon(a)$. Letting $\epsilon \to 0$, we obtain the existence of a solution u^* in G which satisfies

$$\tilde{u}_{k,a}^{H^1} \leqslant u^* \leqslant v \quad \text{in } G, \tag{2.32}$$

where we denote by $\tilde{u}_{k,a}^{H^1}$ the extension of $u_{k,a}^{H^1}$ by zero in $\overline{H^1}^c$. We conclude that (2.30) holds in H^1. In order to extend this convergence to whole G, we proceed as in the proof of Proposition 2.3, with a minor modification due to the geometry. We put the origin of coordinates at a, takes the same spherical coordinates and obtain again that

$$r^2 u_r^* \mathbf{i} - r u_\phi^* \mathbf{e} + \frac{r}{\sin\phi} \nabla_{\sigma'} u^* \to -2k\sigma_N \mathbf{i} + k\mathbf{e} \quad \text{uniformly on } S_+^{N-1} \text{ as } r \to 0.$$

Therefore (2.20) holds for any $\phi \in [0,\pi/2]$. For $r > 0$, the angle ϕ ranges from $\psi(r) = \cos^{-1}(-r/2)$ to 0 (here is the difference with the half-space case) and $|x|^2 \nabla u(x)$ remains bounded in this domain, by the regularity theory for quasilinear elliptic equations. Since

$$u^*(r,\sigma',\phi)=\int_{\psi(r)}^{\phi} u^*_\phi(r,\sigma',\theta)\,d\theta, \tag{2.33}$$

we derive, as in the proof of Proposition 2.3,

$$\lim_{r\to 0} u^*(r,\sigma',\phi)=k\cos\phi \quad \text{uniformly on } [0,\pi/2]. \tag{2.34}$$

The proof that (2.30) holds is a particular case of Theorem 2.7. $\square$

In a general domain we have to extend the solution through the boundary. We denote by $\dot\rho(x)$ the signed distance from $x\to\partial\Omega$, that is $\dot\rho(x)=\rho(x)$ if $x\in\Omega$ and $\dot\rho(x)=-\rho(x)$ if $x\in\Omega^c$. Since $\partial\Omega$ is C^2, there exists $\beta_0>0$ such that if $x\in\mathbb{R}^N$ verifies $-\beta_0\leqslant\dot\rho(x)\leqslant\beta_0$, there exists a unique $\xi_x\in\partial\Omega$ such that $|x-\xi_x|=|\dot\rho(x)|$. Furthermore, if ν_{ξ_x} is the outward unit vector to $\partial\Omega$ at ξ_x, $x=\xi_x-\dot\rho(x)\nu_{\xi_x}$. In particular, $\xi_x-\dot\rho(x)\nu_{\xi_x}$ and $\xi_x+\dot\rho(x)\nu_{\xi_x}$ have the same orthogonal projection ξ_x onto $\partial\Omega$.

Let $T_{\beta_0}(\Omega)=\{x\in\mathbb{R}^N\colon -\beta_0\leqslant\dot\rho(x)\leqslant\beta_0\}$, then the mapping $\Pi:[-\beta_0,\beta_0]\times\partial\Omega\mapsto T_{\beta_0}(\Omega)$ defined by $\Pi(\rho,\xi)=\xi-\dot\rho\nu(\xi)$ is a C^2 diffeomorphism. Moreover, $D\Pi(0,\xi)(1,e)=e-\nu_\xi$ for any e belonging to the tangent space $T_\xi(\partial\Omega)$ to $\partial\Omega$ at ξ. If $x\in T_{\beta_0}(\Omega)$, we define the reflection of x through $\partial\Omega$ by $\psi(x)=\xi_x+\dot\rho(x)\nu_{xi_x}$. Clearly ψ is an involutive diffeomorphism from $\overline{\Omega}\cap T_{\beta_0}(\Omega)$ to $\Omega^c\cap T_{\beta_0}(\Omega)$. Furthermore, for any $\xi\in\partial\Omega$, $D\psi(\xi)=\mathcal{S}_{T_\xi(\partial\Omega)}$ is the symmetry with respect to the tangent space $T_\xi(\partial\Omega)$ to $\partial\Omega$ at ξ. If a function v is defined in $\Omega\cap T_{\beta_0}(\Omega)$, we define $\tilde v$ in $\Omega^c\cap T_{\beta_0}(\Omega)$ by

$$\tilde v(x)=\begin{cases} v(x) & \text{if } x\in\Omega\cap T_{\beta_0}(\Omega),\\ -v\circ\psi(x) & \text{if } x\in\Omega^c\cap T_{\beta_0}(\Omega).\end{cases} \tag{2.35}$$

Proposition 2.6. *Let $v\in C^{1,\alpha}(\overline{\Omega}\cap T_{\beta_0}(\Omega)\setminus\{0\})$ be a solution of (2.1) in $\Omega\cap T_{\beta_0}(\Omega)$ vanishing on $\partial\Omega\setminus\{0\}$. Then $\tilde v\in C^{1,\alpha}(T_{\beta_0}(\Omega)\setminus\{0\})$ is solution of a quasilinear equation*

$$-\sum_j\frac{\partial}{\partial x_j}\tilde A_j(x,D\tilde v)+\tilde b(x)|\tilde v|^{q-1}\tilde v=0 \tag{2.36}$$

in $T_{\beta_0}(\Omega)\setminus\{0\}$ where the $\tilde A_j$ and $\tilde b$ are C^1 functions defined in $T_{\beta_0}(\Omega)$ where they verify

$$\begin{cases} \text{(i)} & \tilde A_j(x,0)=0,\\ \text{(ii)} & \displaystyle\sum_{i,j}\frac{\partial\tilde A_j}{\partial\eta_i}(x,\eta)\xi_i\xi_j\geqslant\Gamma|\eta|^{p-2}|\xi|^2,\\ \text{(iii)} & \displaystyle\sum_{i,j}\left|\frac{\partial\tilde A_j}{\partial\eta_i}(x,\eta)\right|\leqslant\Gamma|\eta|^{p-2},\\ \text{(iv)} & \Gamma\geqslant\tilde b(x)\geqslant\gamma\end{cases} \tag{2.37}$$

for all $x\in T_\beta(\Omega)\setminus\{0\}$ for some $\beta\in(0,\beta_0]$, $\eta\in\mathbb{R}^N$, $\xi\in\mathbb{R}^N$ and some $0<\gamma\leqslant\Gamma$.

Proof. Assumptions (2.37) implies that weak solutions of (2.36) are $C^{1,\alpha}$, for some $\alpha > 0$ [17] and satisfy the standard a priori estimates. As it is defined the function $\tilde{v}$ is clearly C^1 in $T_{\beta_0}(\Omega)\setminus\{0\}$. Writing $Dv(x) = -D(\tilde{v}\circ\psi(x)) = -D\psi(x)(D\tilde{v}(\psi(x)))$ and $\tilde{x} = \psi(x) = \psi^{-1}(x)$:

$$\int\limits_{\Omega\cap T_{\beta_0}(\Omega)} \big(|Dv|^{p-2}Dv\,.\,D\zeta + |v|^{q-1}v\zeta\big)\,dx$$
$$= \int\limits_{\overline{\Omega}^c\cap T_{\beta_0}(\Omega)} \big(\big|D\psi(D\tilde{v})\big|^{p-2} D\psi(D\tilde{v})\,.\,D\psi(D\zeta) + |\tilde{v}|^{q-1}\tilde{v}\zeta\big(\psi(\tilde{x})\big)\big)|D\psi|\,d\tilde{x}.$$

But

$$D\psi(D\tilde{v})\,.\,D\psi(D\zeta) = \sum_k\bigg(\sum_i \frac{\partial\psi_i}{\partial x_k}\frac{\partial\tilde{v}}{\partial x_i}\bigg)\bigg(\sum_j \frac{\partial\psi_j}{\partial x_k}\frac{\partial\zeta}{\partial x_j}\bigg)$$
$$= \sum_j\bigg(\sum_{i,k} \frac{\partial\psi_i}{\partial x_k}\frac{\partial\psi_j}{\partial x_k}\frac{\partial\tilde{v}}{\partial x_i}\bigg)\frac{\partial\zeta}{\partial x_j}.$$

We set $b(x) = |D\psi|$,

$$A_j(x,\eta) = |D\psi|\big|D\psi(\eta)\big|^{p-2}\sum_i\bigg(\sum_k \frac{\partial\psi_i}{\partial x_k}\frac{\partial\psi_j}{\partial x_k}\bigg)\eta_i, \tag{2.38}$$

and

$$A(x,\eta) = \big(A_1(x,\eta),\dots,A_N(x,\eta)\big) = |D\psi|\big|D\psi(\eta)\big|^{p-2}(D\psi)^t D\psi(\eta). \tag{2.39}$$

For any $\xi\in\partial\Omega$, the mapping $D\psi_{\partial\Omega}(\xi)$ is the symmetry with respect to the hyperplane $T_\xi(\partial\Omega)$ tangent to $\partial\Omega$ at ξ, so $|D\psi(\xi)| = 1$. Inasmuch $D\psi$ is continuous, a lengthy but standard computation leads to the existence of some $\beta\in(0,\beta_0]$ such that (2.37) holds in $T_\beta(\Omega)\cap\overline{\Omega}^c$. If we define $\tilde{A}$ (respectively $\tilde{b}$) to be $|\eta|^{p-2}\eta$ (respectively 1) on $T_\beta(\Omega)\cap\overline{\Omega}$ and A (respectively $|D\psi|$) on $T_\beta(\Omega)\cap\overline{\Omega}^c$, then inequalities (2.37) are satisfied in $T_\beta(\Omega)$. □

Remark. Notice that, similarly to the p-Laplacian, the vector field $\tilde{A}$ is positively homogeneous with exponent $p-1$ with respect to η. Furthermore, if for $r>0$ we set $\tilde{A}^r_j(x,\eta) = \tilde{A}_j(rx,\eta)$, then $\tilde{A}^r_j$ satisfies the same estimates (2.37) as A_j, uniformly in $T_{r^{-1}\beta}(r^{-1}\Omega)$, for $0<r\leqslant 1$. Furthermore,

$$\lim_{r\to 0} A^r_j(x,\eta) = |\eta|^{p-2}\eta_j \quad \forall\eta\in\mathbb{R}^N,\ \forall j = 1,\dots,N,$$

and this limit is uniform on the bounded subsets of $\mathbb{R}^N$.

Theorem 2.7. *Let Ω be a bounded domain with a C^2 boundary and $a\in\partial\Omega$. Assume $N-1<q<2N-1$ and denote by $\rho(x)$ the distance from x to $\partial\Omega$. Then for any $k>0$ there exists* *a unique function $u = u_{k,a} \in C(\overline{\Omega} \setminus \{a\})$ which vanishes on $\partial\Omega \setminus \{a\}$, is solution of* (2.1) *and satisfies*

$$u_{k,a}(x) = k\frac{\rho(x)}{|x-a|^2}\bigl(1+\mathrm{o}(1)\bigr) \quad \textit{as } x \to a. \tag{2.40}$$

Proof. Uniqueness follows from (2.40) by the same technique as in the previous propositions. For existence let B^i_R be a ball of radius R such that $B^i_R \subset \Omega$ and $a \in \partial B^i_R$, and let ω_i be its center. We denote by U^i the solution of (2.1) in B^i_R, which vanishes on $\partial B^i_R \setminus \{a\}$ and satisfies

$$U^i(x) = k\frac{R-|x-\omega_i|}{|x-a|^2}\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } x \to a. \tag{2.41}$$

If we set $U_\delta = \eta_\delta(U^i)$, we have already seen that $\check{U}_\delta$, the extension of U_δ by zero outside its support, is a subsolution of (2.1) in Ω. Because V^Ω_a, the N-harmonic function element of $C(\overline{\Omega} \setminus \{a\})$ vanishing on $\partial\Omega \setminus \{a\}$, satisfies

$$V^\Omega_a(x) = \frac{\rho(x)}{|x-a|^2}\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } x \to a,\ x \in B^i_R, \tag{2.42}$$

there holds $kV^\Omega_a \geqslant \check{U}_\delta$. If $\Omega_\epsilon = \Omega \setminus \{B_\epsilon(a)\}$ ($\epsilon > 0$), we construct a solution $u_\epsilon \in C(\overline{\Omega_\epsilon})$ of (2.1) in Ω_ϵ, which vanishes on $\partial\Omega \setminus B_\epsilon(a)$ and takes the value kV^Ω_a on $\partial B_\epsilon(a) \cap \Omega$. By the maximum principle $\epsilon \mapsto u_\epsilon$ is increasing and $\check{U}_\delta \leqslant u_\epsilon \leqslant kV^\Omega_a$ in Ω_ϵ. Letting $\epsilon \to 0$ we obtain that u_ϵ converges in the C^1_{loc}-topology of $\overline{\Omega} \setminus \{a\}$ to a solution $u = u_{k,a}$ of (2.1) in Ω. It follows from the previous inequalities that

$$\check{U}_\delta(x) \leqslant u(x) \leqslant kV^\Omega_a(x) \quad \forall x \in \overline{\Omega} \setminus \{a\}. \tag{2.43}$$

In order to prove the asymptotic behaviour, we proceed as in Proposition 2.4 with the help of the reflection principle of Proposition 2.6. We fix the origin of coordinates at $a = 0$ and the normal outward unit vector at a to be $-\mathbf{e}_N$. If $\tilde{u}$ is the extension of u by reflection through $\partial\Omega$, it satisfies

$$-\sum_j \frac{\partial}{\partial x_j}\tilde{A}_j(x, D\tilde{u}) + \tilde{b}(x)|\tilde{u}|^{q-1}\tilde{u} = 0 \tag{2.44}$$

in $T^\beta(\Omega) \setminus \{0\}$. For $r > 0$, set $\tilde{u}^r(x) = r\tilde{u}(rx)$. Then $\tilde{u}^r$ is solution of

$$-\sum_j \frac{\partial}{\partial x_j}\tilde{A}^r_j\bigl(x, D\tilde{u}^r\bigr) + r^{2N-1-q}\tilde{b}(rx)\bigl|\tilde{u}^r\bigr|^{q-1}\tilde{u}^r = 0 \tag{2.45}$$

in $T^{\beta r^{-1}}(\Omega^r) \setminus \{0\}$, where $\Omega^r := r^{-1}\Omega$. By [3, Theorem 2.4] there exists $C > 0$ such that

$$kV^\Omega_0(x) \leqslant Ck\frac{\rho(x)}{|x|^2}.$$

Furthermore, for any $x \in T^\beta(\Omega) \setminus \{0\}$, $\rho(x) := \operatorname{dist}(x, \Omega) = \rho(\psi(x))$ (we recall that $\psi(x)$ is the symmetric of x with respect to $\partial\Omega$ as it is defined in Proposition 2.6), and $c|x| \leqslant |\psi(x)| \leqslant$

$c^{-1}|x|$ for some $c > 0$, the same relations holds if $T^{\beta}(\Omega)$ is replaced by $T^{\beta r^{-1}}(\Omega^r)$ and $\rho(x)$ by $\rho_r(x) := \operatorname{dist}(x, \Omega^r)$. Since Ω is C^2,

$$\lim_{r\to 0} \frac{\rho(rx)}{r\rho_r(x)} = 1$$

uniformly on bounded subsets of $\mathbb{R}^N$. Consequently

$$\left|\tilde{u}^r\right|(x) \leqslant Ckr^{-1}\frac{\rho(rx)}{|x|^2} = Ck\frac{\rho_r(x)}{|x|^2}\bigl(1+\mathrm{o}(1)\bigr).$$

For $0 < a < b$ fixed and $r \leqslant r_0$ (for some $r_0 \in (0,1]$) the spherical shall $\Gamma_{a,b} = \{x \in \mathbb{R}^N\colon a \leqslant |x| \leqslant b\}$ is included into $T^{\beta r^{-1}}(\Omega^r)$. By the classical regularity theory for quasilinear equations [17] and Proposition 2.6, there holds

$$\left\| D\tilde{u}^r \right\|_{C^\alpha(\Gamma_{2/3,3/2})} \leqslant C_r \left\| \tilde{u}^r \right\|_{L^\infty(\Gamma_{1/2,2})}, \tag{2.46}$$

where C_r remains bounded because $r \leqslant 1$. By Ascoli's theorem and (2.43) $\tilde{u}^r(x)$ converges to $kx_N|x|^{-2}$ in the $C^1(B_{3/2} \setminus B_{1/2})$-topology. This implies, in particular,

$$\lim_{r\to 0} r^2 D\tilde{u}(rx) = -2kx_N x|x|^{-4} + k|x|^{-2}\mathbf{e}_N.$$

If we take, in particular, $|x| = 1$, we derive

$$\lim_{r\to 0}\bigl(r\tilde{u}(r,\sigma), r^2\nabla\tilde{u}(r,\sigma)\bigr) = (k\cos\phi, -k\sin\phi\,\mathbf{e}_N), \tag{2.47}$$

uniformly with respect to $\sigma = (\sin\phi\sigma', \cos\phi) \in S^{N-2} \times [0,\pi]$. Because $\partial\Omega$ is C^2, there exists $\epsilon_0 > 0$ and a C^2 real-valued function h defined in $\Theta_{\epsilon_0} := B_{\epsilon_0} \cap \partial H$ (we recall that $\partial H = \{x = (x', 0)\}$) and an open neighborhood $\mathcal{V}_{\epsilon_0}$ of 0 such that $\partial\Omega \cap \mathcal{V}_{\epsilon_0} = \{x = (x', x_N\colon x_N = h(x')\}$, and $Dh(0) = 0$ (this expresses the fact that $\partial H = T_0(\partial\Omega)$). If we define Ψ by

$$\Psi(x) = \bigl(x', x_N - h(x')\bigr) \quad \forall x \in \mathcal{V}_{\epsilon_0}$$

then $\det(D\Psi) = 1$ and $D\Psi(0) = I$. Up to replacing ϵ_0 by a smaller quantity, Ψ is a C^2 diffeomorphism from $\mathcal{V}_{\epsilon_0}$ into a neighborhood $\mathcal{V}'$ of 0 such that $\mathcal{V}_{\epsilon_0} \cap \partial\Omega = \Theta_{\epsilon_0}$. Because $\operatorname{dist}(\Psi(x), \partial H) = x_N - h(x')$, $\operatorname{dist}(\Psi(x), \partial H) = \rho(x)(1 + \mathrm{o}(1))$ as $x \to 0$. Thus, if we set $x = \Psi^{-1}(y)$ and $\tilde{u}(x) = u^*(y)$, (2.47) is equivalent to

$$\lim_{|y|\to 0}\bigl(|y|u^*\bigl(|y|,\sigma\bigr), |y|^2\nabla u^*\bigl(|y|,\sigma\bigr)\bigr) = (k\cos\phi, -k\sin\phi\,\mathbf{e}_N), \tag{2.48}$$

uniformly on S^{N-1}, thus

$$|y|u^*\bigl(|y|,\sigma\bigr) = k\sin\phi\bigl(1+\mathrm{o}(1)\bigr) \quad \text{as } |y| \to 0 \tag{2.49}$$

uniformly with respect to $\sigma \in S_+^{N-1}$, because u^* vanishes on $B_{\epsilon_0} \cap \partial H \setminus \{0\}$. This implies (2.40). $\square$

Clearly the mapping $k \mapsto u_{k,a}$ is increasing. As u_k satisfies the estimates (2.7) and (2.30), $u_{k,a}$ converges in the $C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{a\})$-topology, as $k \to \infty$, to some $u_{\infty,a}$, solution of (2.1) in Ω, vanishes on $\partial\Omega \setminus \{a\}$ and satisfies

$$\lim_{x \to a} \frac{|x-a|^2 u_{\infty,a}(x)}{\rho(x)} = \infty. \tag{2.50}$$

In order to describe the precise behaviour of $u_{\infty,a}$, we have to introduce separable solutions of (2.1) in $\mathbb{R}^N \setminus \{0\}$: if we look for solutions u under the form $u(r,\sigma) = r^\beta \omega(\sigma)$, then $\beta = -\beta_q = -N/(q+1-N)$ and ω satisfies

$$-\operatorname{div}_\sigma\left(\left(\beta_q^2\omega^2 + |\nabla_\sigma \omega|^2\right)^{(N-2)/2}\nabla_\sigma \omega\right) - \Lambda\left(\beta_q^2\omega^2 + |\nabla_\sigma \omega|^2\right)^{(N-2)/2}\omega + |\omega|^{q-1}\omega = 0 \tag{2.51}$$

on S^{N-1} where $\Lambda = (N-1)\beta_q^2$. We shall denote by $\mathcal{S}_q$ the set of (always $C^{1,\alpha}$) solutions of (2.51). If u is a separable solution of (2.1) in H which vanishes on $\partial H \setminus \{0\}$, the function ω is a solution of (2.51) in S^{N-1}_+ which vanishes on $\partial S^{N-1}_+ = S^{N-2}$. We shall denote by $\mathcal{S}^*_q$ the set of such functions and by $\mathcal{S}^*_{q+}$ the subset of positive solutions. We recall some simple facts.

Proposition 2.8.

(i) *For any $q > N-1$, $\mathcal{S}_q$ contains at least the three constant functions*:

$$0 \quad \text{and} \quad \pm\left((N-1)\beta_q^N\right)^{1/(q+1-N)}.$$

(ii) *For any $q \geqslant 2N-1$, $\mathcal{S}^*_q = \{0\}$.*

(iii) *For any $q \in (N-1, 2N-1)$, $\mathcal{S}^*_{q+}$ contains a unique element.*

Proof. Assertion (i) is evident since $\Lambda > 0$. Assertion (ii), as well as the existence part of assertion (iii), can be found in [8,21]. Furthermore, any $\omega \in \mathcal{S}^*_{q+}$ is positive in S^{N-1}_+ and verifies $\omega_\phi < 0$ by Hopf boundary lemma as the outward normal derivative on ∂S^{N-1}_+ is $\partial/\partial\phi$. We can construct a minimal element in $\mathcal{S}^*_{q+}$ in the following way. If we denote by u^H_k the unique solution of (2.1) in H which satisfies (2.10) and set $T_r(u^H_k)(x) = r^{\beta_q} u^H_k(rx)$ for $r > 0$, then $T_r(u^H_k)$ is a solution of (2.1) in H which satisfies

$$T_r\left(u^H_k\right) = r^{(2N-1-q)/(q+1-N)} k \frac{x_N}{|x|^2}\left(1 + \mathrm{o}(1)\right) \quad \text{as } x \to 0.$$

Thus $T_r(u^H_k) = u^H_{r^{(2N-1-q)/(q+1-N)}k}$. Furthermore, if $\omega \in \mathcal{S}^*_{q+}$, the maximum principle at 0 and at infinity (replacing u_ω by $u_\omega + \epsilon$ and letting $\epsilon \to 0$) leads to

$$u_\omega(r,\sigma) := r^{-\beta_q}\omega(\sigma) > u^H_k(r,\sigma) \quad \forall (r,\sigma) \in (0,\infty) \times S^{N-1}_+,\ \forall k > 0.$$

Letting $k \to \infty$ implies $u_\omega(r,\sigma) \geqslant u^H_\infty(r,\sigma)$ and $T_r(u^H_\infty) = u^H_\infty$ given that $2N-1-q > 0$. Then the function u^H_∞ is invariant with respect to the transformation T_r. It is therefore self-similar, and consequently under the form $u^H_\infty(r,\sigma) = r^{-\beta_q}\underline{\omega}(\sigma)$. As a result of the previous inequality $\underline{\omega}$ is the minimal element of $\mathcal{S}^*_{q+}$. Next we denote $\delta^* = \max\{\delta \geqslant 0\colon \delta\omega \leqslant \underline{\omega}\}$ and $u_{\omega,\delta^*} = \delta^* u_\omega$. Notice that $\delta^* \in (0,1]$ as $\underline{\omega} > 0$ in S^{N-1}_+ and satisfies Hopf boundary lemma on ∂S^{N-1}_+. Clearly

u_{ω,δ^*} is a subsolution for (2.1) and it is dominated by u_∞^H in H. Furthermore, $\delta^*\omega \leqslant \underline{\omega}$ in $\overline{S_+^{N-1}}$, $\delta^*\omega_\phi \leqslant \underline{\omega}_\phi$ on ∂S_+^{N-1}, and

(i) either there exists $\sigma_0 \in S_+^{N-1}$ such that $\delta^*\omega(\sigma_0) = \underline{\omega}(\sigma_0)$,
(ii) or $\delta^*\omega < \underline{\omega}$ in S_+^{N-1} and there exists $\sigma_0' \in S^{N-2}$ such that $\delta^*\omega_\phi(\sigma_0', \pi/2) = \underline{\omega}_\phi(\sigma_0', \pi/2)$.

In case (i), and as Du_∞^H never vanishes in H, it follows from [6, Lemma 1.3] (a variant of the strong comparison principle) that $u_{\omega,\delta^*} = \underline{u}$. This implies that u_{ω,δ^*} is a solution, $\delta^* = 1$ and, consequently $\omega = \underline{\omega}$.

In case (ii) we follow the linearization procedure already introduced in [6]. By the mean value theorem

$$\left|Du_\infty^H\right|^{N-2} u_{\infty x_i} - |Du_{\omega,\delta^*}|^{N-2} u_{\omega,\delta x_i} = \sum_j \alpha_{ij}\left(u_\infty^H - u_{\omega,\delta^*}\right)_{x_j},$$

where

$$\begin{aligned}\alpha_{ij} = {} & \left|t_i Du_\infty^H + (1-t_i)Du_{\omega,\delta^*}\right|^{N-4}\Big(\delta_{ij}\left|t_i Du_\infty^H + (1-t_i)Du_{\omega,\delta^*}\right|^2 \\ & + (N-2)\big(t_i u_{\infty x_i}^H + (1-t_i)u_{\omega,\delta^* x_i}\big)\big(t_i u_{\infty x_j}^H + (1-t_i)u_{\omega,\delta^* x_j}\big)\Big),\end{aligned}$$

with $0 \leqslant t_i \leqslant 1$. Next $w = u_\infty^H - u_{\omega,\delta^*}$ is positive in H and satisfies

$$-\sum_{ij}(\alpha_{ij} w_{x_j})_{x_i} + cw \geqslant 0,$$

where $c = ((u_\infty^H)^q - u_{\omega,\delta^*}^q)/(u_\infty^H - u_{\omega,\delta^*}) > 0$. Notice that $(\alpha_{ij}(x))$ is the Hessian of a strictly convex function therefore it is nonnegative and that $(\alpha_{ij})(r, \sigma_0', \pi/2)$ is positive-definite. Therefore it is positive-definite in a neighborhood of $(r, \sigma_0', \pi/2)$ (independent of r, actually). Inasmuch $(u_\infty^H - u_{\omega,\delta^*})_{x_N} = 0$ at $(r, \sigma_0', \pi/2)$, we derive a contradiction with Hopf lemma. Therefore case (ii) cannot occur and $\omega = \underline{\omega}$. □

Remark. If we look for separable solutions of

$$-\operatorname{div}\big(|Du|^{p-2}Du\big) + |u|^{q-1}u = 0, \tag{2.52}$$

in $\mathbb{R}^N$, where $q > p-1 > 0$, p not necessarily equal to N or to 2, under the form $u(r,\sigma) = r^\beta\omega(\sigma)$, then $\beta = \beta_{p,q} = -p/(q+1-p)$ and ω is a solution of

$$-\operatorname{div}_\sigma\Big(\big(\beta_{p,q}^2\omega^2 + |\nabla_\sigma\omega|^2\big)^{(p-2)/2}\nabla_\sigma\omega\Big) - \Lambda(p,q)\big(\beta_{p,q}^2\omega^2 + |\nabla_\sigma\omega|^2\big)^{(p-2)/2}\omega + |\omega|^{q-1}\omega = 0 \tag{2.53}$$

on S^{N-1} where $\Lambda(p,q) = \beta_{p,q}^{p-1}(q\beta_{p,q} - p)$. If we look for separable solutions in H which vanishes on $\partial H \setminus \{0\}$ the solution ω of (2.53) is subject to the boundary condition $\omega = 0$ on $\partial S_+^{N-1} = S^{N-2}$. A fairly exhaustive theory of existence is developed in [8,21]. The existence of non-trivial solution of (2.53) is insured as soon as $\Lambda(p,q) > 0$, or equivalently $q < N(p-1)/$

$(N-p)$ if $p<N$, and no condition if $p \geqslant N$. If $q \geqslant N(p-1)/(N-p)$ no solution exists, up to the trivial one. This is linked to the removability result proved by Vàzquez and Véron [18]. The existence of non-trivial solutions of the same equation in S_+^{N-1} vanishing on ∂S_+^{N-1} is much more complicated. However, it is proved in [8,21] that there exists a critical exponent $q_{\mathrm{c}} > p-1$ such that, if $q \geqslant q_{\mathrm{c}}$ no non-trivial solution exists while if $p-1<q<q_{\mathrm{c}}$ there exists a unique positive solution in S_+^{N-1} vanishing on ∂S_+^{N-1}. The uniqueness proof in the previous proposition is valid.

The next result characterizes the solution of (2.1) with a strong singularity on the boundary. In order to express the result, we assume that the outward normal unit vector to $\partial\Omega$ at a is $-\mathbf{e}_N$.

Theorem 2.9. *Let Ω be a bounded domain with a C^2 boundary and $a \in \partial\Omega$. Assume $0<p-1<q<2N-1$. Then there exists a unique function $u \in C^1(\overline{\Omega} \setminus \{a\})$ which vanishes on $\partial\Omega \setminus \{a\}$, is solution of* (2.1) *in Ω and satisfies*

$$\lim_{x \to a} \frac{|x-a|^2 u(x)}{\rho(x)} = \infty. \tag{2.54}$$

Furthermore,

$$\lim_{\substack{x \to a \\ (x-a)/|x-a| \to \sigma}} |x-a|^{\beta_q} u(x) = \omega(\sigma), \tag{2.55}$$

locally uniformly on S_+^{N-1}. Finally $u = u_{\infty,a} = \lim_{k \to \infty} u_{k,a}$.

Proof. We already know that $u_{\infty,a}$ satisfies (2.54). By translation we fix the origin 0 of coordinates at the point a and we assume that $-\mathbf{e}_N$ is the outward unit vector to $\partial\Omega$ at 0. If G is any C^2 domain in $\mathbb{R}^N$ to the boundary of which 0 belongs, we denote by u_k^G the solution of (2.1) in G, which vanishes on $\partial G \setminus \{0\}$ and verifies

$$u_k^G = k \frac{\rho_G(x)}{|x|^2} \big(1 + \mathrm{o}(1)\big) \quad \text{as } x \to 0, \tag{2.56}$$

where $\rho_G(x) = \operatorname{dist}(x, G)$. When there is no ambiguity, $u_k^\Omega = u_k$. By the maximum principle $G \subset G'$ implies $u_k^G \leqslant u_k^{G'}$ in G. By dilation we can assume that there exist two balls of radius 1, $B \subset \Omega$ and $B' \subset \overline{\Omega}^c$ with respective center $b = \mathbf{e}_N$ and $b' = -b$ with the property that $0 = \partial B \cap \partial B'$. It follows from the maximum principle, the fact that $u_k^B(x) = u_k^{B'}(\mathcal{S}(x))$ where $\mathcal{S}$ is the symmetry with respect to the hyperplane ∂H and Propositions 2.4, 2.5 that

$$\begin{aligned}
&\text{(i)} \quad u_k^B(x) \leqslant u_k(x) \leqslant u_k^{B'^c}(x) \leqslant u_k^{B'}\left(b' + \frac{x-b'}{|x-b'|^2}\right) = u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B, \\
&\text{(ii)} \quad u_k(x) \leqslant u_k^{B'^c}(x) \leqslant u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in \Omega,
\end{aligned} \tag{2.57}$$

and similarly

$$\text{(i)}\quad u_k^B(x) \leqslant u_k^H(x) \leqslant u_k^{B'^c}(x) \leqslant u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B,$$
$$\text{(ii)}\quad u_k^H(x) \leqslant u_k^{B'^c}(x) \leqslant u_k^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in H. \tag{2.58}$$

Letting $k \to \infty$, we obtain

$$\text{(i)}\quad u_\infty^B(x) \leqslant u_\infty(x) \leqslant u_\infty^{B'^c}(x) \leqslant u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B,$$
$$\text{(ii)}\quad u_\infty(x) \leqslant u_\infty^{B'^c}(x) \leqslant u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in \Omega, \tag{2.59}$$

as well as

$$\text{(i)}\quad u_\infty^B(x) \leqslant |x|^{-\beta_q}\omega\left(\frac{x}{|x|}\right) \leqslant u_\infty^{B'^c}(x) \leqslant u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in B,$$
$$\text{(ii)}\quad |x|^{-\beta_q}\omega\left(\frac{x}{|x|}\right) \leqslant u_\infty^{B'^c}(x) \leqslant u_\infty^B\left(\mathcal{S}\left(b' + \frac{x-b'}{|x-b'|^2}\right)\right) \quad \forall x \in H. \tag{2.60}$$

From (2.60)(i) and the fact that $b' = -b$, we also derive

$$|x|^{-\beta_q}\omega\big(x/|x|\big) \leqslant u_\infty^{B'^c}(x) \leqslant \left|\mathcal{S}\left(\frac{x+b}{|x+b|^2} - b\right)\right|^{-\beta_q} \omega\left(\frac{\mathcal{S}(x+b-|x+b|^2 b)}{|\mathcal{S}(x+b-|x+b|^2 b)|}\right). \tag{2.61}$$

But

$$\left|\mathcal{S}\left(\frac{x+b}{|x+b|^2} - b\right)\right| = \frac{|x|}{|x+b|} = |x|\big(1+\mathrm{o}(1)\big) \quad \text{as } x \to 0$$

(remember that $|b| = 1$). If $x = (x_1, \dots, x_N)$, $|x+b|^2 = |x|^2 + 1 + 2x_N$ and

$$\mathcal{S}\big(x+b-|x+b|^2 b\big) = \big(x_1, \dots, x_N + |x|^2\big).$$

Thus (2.61) becomes

$$|x|^{-\beta_q}\omega\big(x/|x|\big) \leqslant u_\infty^{B'^c}(x) \leqslant |x|^{-\beta_q}|x+b|^{\beta_q}\omega\left(\frac{x+|x|^2\mathbf{e}_N}{|x|\sqrt{1+|x|^2+2x_N}}\right). \tag{2.62}$$

If we assume $|x|^2 = \mathrm{o}(x_N)$ then

$$\big(x+|x|^2\mathbf{e}_N\big)/\Big(|x|\sqrt{1+|x|^2+2x_N}\Big) = x\big(1+\mathrm{o}(1)\big)/|x|$$

as $x \to 0$, and

$$u_\infty^{B'^c}(x) = |x|^{-\beta_q}\omega\big(x/|x|\big)\big(1+\mathrm{o}(1)\big). \tag{2.63}$$

If we define $\mathcal{T}$ by

$$\mathcal{T}(x)=\mathcal{S}\left(\frac{x+b}{|x+b|^2}-b\right),$$

then (2.60)(i) reads also as

$$\left|\mathcal{T}^{-1}(x)\right|^{-\beta_q}\omega\left(\frac{\mathcal{T}^{-1}(x)}{|\mathcal{T}^{-1}(x)|}\right)\leqslant u_\infty^B(x)\leqslant |x|^{-\beta_q}\omega\left(\frac{x}{|x|}\right). \tag{2.64}$$

Furthermore,

$$\mathcal{T}^{-1}(x)=\left(\frac{x_1}{|x-b|^2},\dots,\frac{x_{N-1}}{|x-b|^2},\frac{1-x_N}{|x-b|^2}-1\right)=\frac{x-|x|^2\mathbf{e}_N}{|x-b|^2}.$$

Then

$$\left|\mathcal{T}^{-1}(x)\right|=\left|b+\frac{x-b}{|x-b|^2}\right|=\frac{|x|}{|x-b|},$$

and

$$\left|\mathcal{T}^{-1}(x)\right|^{-\beta_q}\omega\left(\frac{\mathcal{T}^{-1}(x)}{|\mathcal{T}^{-1}(x)|}\right)=|x|^{-\beta_q}|x-b|^{\beta_q}\omega\left(\frac{x-|x|^2\mathbf{e}_N}{|x||x-b|}\right).$$

If we assume again $|x|^2=\mathrm{o}(x_N)$ then $(x-|x|^2\mathbf{e}_N)/(|x|\sqrt{1+|x|^2-2x_N})=x(1+\mathrm{o}(1))/|x|$ as $x\to 0$, and

$$u_\infty^B(x)=|x|^{-\beta_q}\omega\big(x/|x|\big)\big(1+\mathrm{o}(1)\big). \tag{2.65}$$

Combining (2.59)(i), (2.62) and (2.64) we obtain that

$$u_\infty(x)=|x|^{-\beta_q}\omega\big(x/|x|\big)\big(1+\mathrm{o}(1)\big)\quad \text{as } x\to 0 \tag{2.66}$$

uniformly on any subset of Ω such that $|x|^2=\mathrm{o}(x_N)$ near 0. In order to obtain the precise behaviour (2.55), we proceed and in the proof of Theorem 2.7. We extend u by reflection through $\partial\Omega$ near 0 and denote by $\tilde{u}$ the extended function defined in $T^\beta(\Omega)$. For $r\in(0,1]$ we define

$$w_r:=T_r(\tilde{u})(x)=r^{\beta_q}\tilde{u}(rx).$$

Then w_r satisfies

$$-\sum_j\frac{\partial}{\partial x_j}\tilde{A}_j^r(x,Dw_r)+\tilde{b}(rx)|w_r|^{q-1}w_r=0 \tag{2.67}$$

in $T^{\beta r^{-1}}(\Omega^r)$. Since w_r is uniformly bounded on $\Gamma_{1/2,2}$ (by Proposition 2.1 applied to u and $-u$ and the definition of the reflected function), $Dw_r(u)$ is bounded in $C^\alpha(\Gamma_{2/3,3/2})$. By Ascoli's theorem w_r converges in the $C^1(\Gamma_{2/3,3/2})$-topology to $x\mapsto |x|^{-\beta_q}\tilde{\omega}(x/|x|)$, where $\tilde{\omega}$ is defined from ω by reflection through the equator ∂S_+^{N-1}. In order to get rid of the boundary, we use again the C^2 diffeomorphism Ψ which sends B_{ϵ_0} onto itself and verifies $\Psi(B_{\epsilon_0}\cap\partial\Omega)=B_{\epsilon_0}\cap\partial H$. We set $x=\Psi^{-1}(y)$ and $\tilde u(x)=u^*(y)$. Then

$$\lim_{|y|\to 0}\big(|y|^{\beta_q}u^*\big(|y|,\sigma\big),|y|^{\beta_q+1}\nabla u^*\big(|y|,\sigma\big)\big)=\big(\omega(\phi),-\omega_\phi\mathbf{e}_N\big),\tag{2.68}$$

uniformly on S^{N-1}, thus

$$u^*\big(|y|,\sigma\big)=|y|^{\beta_q}\omega(\phi)\big(1+\mathrm{o}(1)\big)\quad\text{as } |y|\to 0\tag{2.69}$$

uniformly with respect to $\sigma\in S_+^{N-1}$, because u^* vanishes on $B_{\epsilon_0}\cap\partial H\setminus\{0\}$. Actually, a stronger result than (2.55) follows, namely,

$$u(x)=|x|^{-\beta_q}\omega\big(x/|x|\big)\big(1+\mathrm{o}(1)\big)\quad\text{as } x\to 0.\tag{2.70}$$

Mutatis mutandis, this estimate implies uniqueness of a solution with a strong singularity as in Theorem 2.7. □

3. The removability result

In this section Ω is a C^2 domain of $\mathbb{R}^N$ and $a\in\partial\Omega$. The next result extends the removability result of Gmira–Véron [7] dealing with semilinear equations.

Theorem 3.1. *Let g be a continuous function defined on $\mathbb{R}$ which satisfies*

$$\liminf_{r\to\infty} g(r)/r^{q_c}>0\quad\textit{and}\quad\limsup_{r\to-\infty} g(r)/|r|^{q_c}<0,\tag{3.1}$$

where $q_c:=2N-1$ and let $u\in C^1(\overline{\Omega}\setminus\{a\})$ be a solution of

$$-\operatorname{div}\big(|Du|^{N-2}Du\big)+g(u)=0\quad\text{in }\Omega\tag{3.2}$$

which coincides with some $\phi\in C^1(\partial\Omega)$ on $\partial\Omega\setminus\{a\}$. Then u extends to $\overline{\Omega}$ as a continuous function.

Proof. Without any loss of generality, we can assume that Ω is bounded, $a=0$ and $-\mathbf{e}_N$ is the outward normal vector to $\partial\Omega$ at 0. We denote by V_0^Ω the solution of (2.2) in Ω which vanishes on $\partial\Omega\setminus\{0\}$ and satisfies

$$V_0^\Omega(x)=\frac{\rho(x)}{|x|^2}\big(1+\mathrm{o}(1)\big)\quad\text{as } x\to 0.$$

Let M be the supremum of $|\phi|$ on $\partial\Omega$ and $\widetilde M=\max\{M,(B/A)^{1/q}\}$. By assumption there exists $A>0$ and $B\geqslant 0$, depending only on g, such that

$$-\operatorname{div}\big(|Du|^{N-2}Du\big)+Au^{q_c}\leqslant B\quad\text{in }\big\{x\in\Omega\colon u(x)>0\big\}.\tag{3.3}$$

If $v = u - \widetilde{M}$, then $v \leqslant 0$ on $\partial\Omega \setminus \{0\}$ and

$$-\operatorname{div}\big(|Dv|^{N-2}Dv\big) + Av^{q_c} \leqslant 0 \quad \text{in } \big\{x \in \Omega\colon v(x) > 0\big\}. \tag{3.4}$$

Using the same functions η_ϵ as in the proof of Proposition 2.1 we deduce that $\eta_\epsilon(v)$ satisfies the same inequality as v, but on whole Ω. By Proposition 2.2 with $q = q_c$ and the expression of V_0^Ω it follows that

$$v(x) \leqslant cV^\Omega(x) \quad \forall x \in \Omega, \tag{3.5}$$

where the constant c depends on A and N. Furthermore, there exists a function $u^* \in C^1(\overline{\Omega} \setminus \{0\})$ such that $0 \leqslant v_+ \leqslant u^*(x) \leqslant cV_0^\Omega$ in Ω, and

$$-\operatorname{div}\big(\big|Du^*\big|^{N-2}Du^*\big) + Au^{*q_c} = 0 \quad \text{in } \Omega. \tag{3.6}$$

As in the proof of Theorem 2.9 we extend u^* through the boundary into $\tilde{u}$ and scale it by setting $T_r(\tilde{u}) := w_r(x) = r\tilde{u}(rx)$ for $r > 0$. Inasmuch all the previous a priori estimates apply (compactness), it follows that there exists a subsequence $\{r_n\}$ converging to 0 and a function $w \in C^1(\mathbb{R}^N \setminus \{0\})$ such that $w_{r_n} \to w$ in the C^1_{loc}-topology of $\mathbb{R}^N \setminus \{0\}$, w is a solution of

$$\begin{cases} -\operatorname{div}(|Dw|^{N-2}Dw) + Aw^{q_c} = 0 & \text{in } \mathbb{R}^N \setminus \{0\}, \\ w \geqslant 0 & \text{in } H = \{x \in \mathbb{R}^N\colon x_N > 0\}, \\ w = 0 & \text{on } \partial H \setminus \{0\}. \end{cases} \tag{3.7}$$

At the end, (3.5) transforms into

$$0 \leqslant w(x) \leqslant c\frac{x_N}{|x|^2} \quad \forall x \in H. \tag{3.8}$$

For $\epsilon > 0$ we denote by W_ϵ the solution of

$$\begin{cases} -\operatorname{div}(|DW_\epsilon|^{N-2}DW_\epsilon) + AW_\epsilon^{q_c} = 0 & \text{in } H \setminus B_\epsilon(0), \\ W_\epsilon = c\epsilon^{-2}x_N & \text{on } H \cap \partial B_\epsilon(0), \\ W_\epsilon = 0 & \text{on } \partial H \setminus B_\epsilon(0). \end{cases} \tag{3.9}$$

By the maximum principle $0 \leqslant w(x) \leqslant W_\epsilon(x) \leqslant cx_N|x|^{-2}$ for any $\epsilon > 0$, and by uniqueness, $T_r(W_\epsilon)(x) = rW_\epsilon(rx) = W_{\epsilon/r}(x)$. Furthermore, $\epsilon \mapsto W_\epsilon$ is increasing. Letting $\epsilon \to 0$ we conclude that W_ϵ decreases to some W_0, which is a solution of

$$\begin{cases} -\operatorname{div}(|DW_0|^{N-2}DW_0) + AW_0^{q_c} = 0 & \text{in } H, \\ W_0 \geqslant 0 & \text{in } H, \\ W_0 = 0 & \text{on } \partial H \setminus \{0\}, \end{cases} \tag{3.10}$$

by the standard regularity results, and satisfies $0 \leqslant w \leqslant W_0$. Finally, W_0 inherits the following scaling invariance property $T_r(W_0)(x) = W_0(x)$ for any $r > 0$. Therefore W_0 is a separable solution which endows the following form:

$$W_0(x) = W_0(r, \sigma) = r^{-1}\omega(\sigma),$$

where ω is nonnegative on S_+^{N-1} and satisfies

$$\begin{cases} -\operatorname{div}_\sigma((\omega^2+|\nabla_\sigma\omega|^2)^{(N-2)/2}\nabla_\sigma\omega) & \\ \quad -(N-1)(\omega^2+|\nabla_\sigma\omega|^2)^{(N-2)/2}\omega+A\omega^{q_c}=0 & \text{in } S_+^{N-1}, \\ \omega=0 & \text{on } \partial S_+^{N-1}. \end{cases} \tag{3.11}$$

By Proposition 2.8, $\omega=0$. Thus $W_0=0 \Rightarrow w=0$, which implies $w_r(x)\to 0$ as $r\to 0$ and equivalently $r\tilde{u}(rx)\to 0$ in the C^1_{loc}-topology of $\mathbb{R}^N\setminus\{0\}$. Consequently $D\tilde{u}(x)=\mathrm{o}(|x|^{-2})$ as $x\to 0$ and finally $u^*(x)=\mathrm{o}(V_0^\Omega(x))$ as $x\to 0$. The maximum principle and the positivity of u^* yields to $u^*\equiv 0$ and finally $u\leqslant \widetilde{M}$ in Ω. In the same way $u\geqslant -\widetilde{M}$. Because the modulus of continuity of u is uniformly bounded near 0, by the classical regularity theory of degenerate elliptic equations (see [12], for example), u extends as a continuous function in whole $\overline{\Omega}$. □

4. The classification theorem

The next result extends some of Gmira–Véron's classification theorem [7, Sections 4, 5] obtained in the study of problem (1.3). In the above mentioned article, the main idea was to reduce the equation to a infinite-dimensional quasi-autonomous evolution system in $\mathbb{R}_+\times S_+^{N-1}$ and to use Lyapounov-energy function. Such an approach cannot be adapted in the quasilinear case. Our method is based upon scaling and uniqueness arguments.

Theorem 4.1. *Assume $N-1<q<2N-1$, Ω is a bounded domain with a C^2 boundary, $a\in\partial\Omega$ and $-\mathbf{e}_N$ is the outward normal unit vector to $\partial\Omega$ at a. Let $u\in C^1(\overline{\Omega}\setminus\{a\})$ be a positive function satisfying* (2.1) *in Ω and vanishing on $\partial\Omega\setminus\{a\}$. Then the following alternative holds.*

(i) *Either there exists $k>0$ such that*

$$u(x)=k\frac{\rho(x)}{|x-a|^2}\big(1+\mathrm{o}(1)\big)\quad \textit{as } x\to a. \tag{4.1}$$

Furthermore, $u=u_{k,a}$, the unique solution of (2.1) *defined in Theorem* 2.7.

(ii) *Or*

$$u(x)=|x-a|^{-\beta_q}\omega(\sigma)\big(1+\mathrm{o}(1)\big)\quad \textit{as } x\to a, \tag{4.2}$$

where ω is the unique positive solution of (2.51) *on S_+^{N-1} which vanishes on ∂S_+^{N-1}, in which case $u=u_{\infty,a}$.*

Proof. We assume that $a=0$ with $\nu_0=-\mathbf{e}_N$ and define

$$k=\limsup_{x\to 0}\frac{u(x)}{V_0^\Omega(x)}=\limsup_{r\to 0}\sup_{|x|=r}\frac{u(x)}{V_0^\Omega(x)}. \tag{4.3}$$

Suppose $k=0$. It follows from the maximum principle that for any $\epsilon>0$ there exists a sequence $r_n\to 0$ such that $0\leqslant u(x)\leqslant \epsilon V_0^\Omega(x)$ in $\Omega\setminus\{B_{r_n}(0)\}$. This fact implies the nullity of u.

Therefore we assume that $k \neq 0$. Assume first that k is finite. Then, for any $\epsilon > 0$, there exists a sequence of points x_n converging to 0 such that

$$\lim_{n\to\infty} \frac{u(x_n)}{V_0^{\Omega}(x_n)} = k \tag{4.4}$$

and

$$\sup_{|x|\leqslant r_n} \frac{u(x)}{V_0^{\Omega}(x)} \leqslant k+\epsilon. \tag{4.5}$$

Since u_k satisfies (2.40) with $a=0$, the two previous relations can be replaced by

$$\begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} \frac{u(x_n)}{u_k(x_n)} = 1, \\ &\text{(ii)} \quad \sup_{|x|\leqslant r_n} \frac{u(x)}{u_k(x)} \leqslant 1+\epsilon. \end{aligned} \tag{4.6}$$

We denote $r_n = |x_n|$, $\xi_n = x_n/r_n$ and define $u_n = r_n u(r_n x)$ and $u_{kn} = r_n u_k(r_n x)$. By the previous arguments combining a priori estimate and regularity theory, there exist a subsequence $\{r_{n_j}\}$ and two nonnegative functions v and v', N-harmonic in H and vanishing on $\partial H \setminus \{0\}$, such that (u_{n_j}, u_{kn_j}) converges to (v, v') in the C^1_{loc}-topology of $H = \mathbb{R}^N_+$. Clearly equality (2.40) implies that $r_{n_j} u_k(r_{n_j} x)$ converges to $kV_0^H(x)$ (which is defined by $kV_0^H(x) := kx_N/|x|^{-2}$) in the same topology. Since v' is uniquely determined by its blow-up at 0, this implies $v' = kV_0^H$ in H. Furthermore, there exists $\xi \in \overline{S^{N-1}_+}$ such that $\xi_{n_k} \to \xi$. If $\xi \in S^{N-1}_+$, $v(\xi) = v'(\xi)$, while, if $\xi \in \partial S^{N-1}_+$, $\partial v/\partial \nu(\xi) = \partial_{x_N} v(\xi) = \partial v'/\partial \nu(\xi)$. In both situation, the tangency conditions of the graphs of v and v' and the strong maximum principle implies that $v = v' = kV_0^H$. By estimate (4.6)(i) and the convergence properties, it follows

$$\lim_{n\to\infty} \frac{u(r_n\xi)}{u_k(r_n\xi)} = 1 \quad \text{uniformly on } |\xi| = 1.$$

Consequently, for any $\delta > 0$, there holds,

$$(1-\delta)u_k(x) \leqslant u(x) \leqslant (1+\delta)u_k(x) \quad \forall x \in \Omega \setminus B_{r_n},$$

for n large enough, which leads to $u_k = u$. At end we consider the case $k = \infty$. Writing (2.1) under the form

$$-\operatorname{div}\big(|Du|^{N-2}Du\big) + d(x)u^{N-1} = 0, \tag{4.7}$$

where $d(x) = |u|^{q+1-N}(x) \leqslant C|x|^{-N}$ by (2.4). We use the boundary Harnack principle. By [1, Theorem 2.2] there exists a constant $c = c(N, q, \Omega) > 0$ such that

$$\frac{1}{c}\frac{u(y)}{\rho(y)} \leqslant \frac{u(x)}{\rho(x)} \leqslant c\frac{u(y)}{\rho(y)} \tag{4.8}$$

for any x and y in Ω such that $|x| = |y|$ be small enough. Since there exists a sequence $x_n \to 0$ such that $\lim_{n\to\infty} u(x_n)/V^\Omega(x_n) \to \infty$, this implies that

$$\lim_{n\to\infty}\left\{\inf \frac{u(x)}{V^\Omega(x)}\colon |x| = |x_n|\right\} = \infty. \tag{4.9}$$

Thus u satisfies (2.54); Theorem 2.9 and (2.55) imply that (4.2) holds. □

The assumption of positivity on u can be weakened if a better a priori estimate is already known. The next result extends [6, Theorem 1.2] into the framework of boundary singularities.

Theorem 4.2. *Assume $N-1 < q < 2N-1$, Ω is a bounded domain with a C^2 boundary, $a \in \partial\Omega$ and $-\mathbf{e}_N$ is the outward normal unit vector to $\partial\Omega$ at a. Let $u \in C^1(\overline{\Omega}\setminus\{a\})$ be a solution of* (2.1) *in Ω vanishing on $\partial\Omega \setminus \{a\}$ such that u/V_a^Ω is bounded in Ω. Then there exists $k \in \mathbb{R}$ such that $u = u_{k,a}$.*

Proof. The outline of the proof are very similar to the finite case of the previous theorem. We still assume $a = 0$ and define k by (4.3). If $k = 0$ the maximum principle implies $u \leqslant 0$ and we return to Theorem 4.1 in the case $u \leqslant 0$. If $k \neq 0$, $k > 0$, for example, (4.4) and (4.5) apply. By the previous scaling method we derive that u_{n_k} converges to some function v in the C^1_{loc}-topology of $H = \mathbb{R}^N_+$ which is N-harmonic in H and vanishes on $\partial H \setminus \{0\}$. Because $r_{n_k} u_k(r_{n_k}x)$ converges to kV_0^H, the tangency condition of v and kV_0^H at some ξ implies that $v = kV_0^H$. Thus $u(x) \geqslant 0$ for $|x| = r_{n_k}$ for n_k large enough. This implies that $u \geqslant 0$ in Ω and we are back to Theorem 4.1. □

Remark. In the semilinear case of problem (1.3), it is proved in [7] that any signed solution u which satisfies $\lim_{x\to a} |x-a|^N u(x) = 0$ has constant sign. The exponent N characterize the minimal changing sign harmonic function vanishing on $\partial\Omega \setminus \{a\}$, with an isolated singularity at a. Changing sign singular N-harmonic functions are constructed in [3]. In particular, there exist singular N-harmonic functions w under the form

$$w(r,\sigma) = r^{-\beta_2}\omega(\sigma),$$

where

$$\beta_2 = \frac{10 - N + \sqrt{81N^2 - 180N + 180}}{10(N-1)}$$

and ω is defined on $S^{N-1}_+ = \{x \in S^{N-1}\colon x_N > 0\}$, vanishes on the equator ∂S^{N-1}_+, is positive on $S^{N-1}_+ \cap \{x\colon x_{N-1} > 0\}$ and negative on $S^{N-1}_+ \cap \{x\colon x_{N-1} < 0\}$. A natural question is therefore whether any signed solution u of (2.1) in Ω which vanishes on $\partial\Omega\setminus\{a\}$ and satisfies $\lim_{x\to a} |x - a|^{\beta_2} u(x) = 0$ has constant sign, and can be henceforth classified through Theorem 4.1.

Final remark. If one replaces the N-harmonic operator by the p-harmonic operator ($p > 1$) and tries to extend the results of Sections 2–4, several difficulties will appear. Even if the existence of separable singular solutions is known, the precise value of the exponent $\beta > 0$ such that $(r,\sigma) \mapsto r^{-\beta}\phi(\sigma)$ is p-harmonic and positive in H and vanishes on $\partial H \setminus \{0\}$ is unknown but for the specific cases $N=2$ or $p=N$ or $p=2$. Notice that in that case the function ϕ satisfies the so-called *spherical p-harmonic spectral equation* (see [9,10,16,19]),

$$\begin{cases} -\operatorname{div}_\sigma((\beta^2\phi^2+|\nabla_\sigma\phi|^2)^{(p-2)/2}\nabla_\sigma\phi)-\lambda(\beta^2\phi^2+|\nabla_\sigma\phi|^2)^{(p-2)/2}\phi=0 & \text{in } S_+^{N-1}, \\ \phi=0 & \text{on } \partial S_+^{N-1}, \end{cases} \tag{4.10}$$

where $\lambda=\beta(\beta(p-1)+p-N)$. If $p=2$ then $\beta=N-1$, while if $N=2$, β is the positive root of the equation

$$3\beta^2+2\frac{p-3}{p-1}\beta-1=0. \tag{4.11}$$

Furthermore, up to now and due to the lack of conformal invariance, it has not been possible to construct the equivalent of the V_a^Ω in a general smooth bounded domain Ω that are positive p-harmonic functions in Ω, vanishing on $\partial\Omega\setminus\{a\}$ and satisfying

$$\lim_{\substack{x\to a\\ (x-a)/|x-a|\to\sigma}} |x-a|^\beta u(x)=\phi(\sigma). \tag{4.12}$$

However, if $\Omega=H=\mathbb{R}_+^N$ the removability and the classification results of Sections 3 and 4 are still valid. The proofs of these theorems are developed in [2].

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